EXERCISE SHEET 2

1) Let X be a complex manifold. Consider the multiplication map

$$\cup : \Lambda(n)_{\mathcal{D}} \otimes \Lambda(m)_{\mathcal{D}} \to \Lambda(n+m)_{\mathcal{D}}$$

Let d denote the differential of the complex $\Lambda(n+m)_{\mathcal{D}}$. For $x \in \Lambda(n)_{\mathcal{D}}$ and $y \in \Lambda(m)_{\mathcal{D}}$, show that

$$d(x \cup y) = dx \cup y + (-1)^{\deg(x)} x \cup dy\,.$$

2) Let $C^\bullet = (C^\bullet, d)$ be a complex of sheaves on a topological space X. Here is another way of looking at hypercohomology: Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover of X and consider the Čech double complex $(\check{C}^\bullet(\mathscr{U}, C^\bullet), d, \delta)$. That is, a the double complex whose i-th column is the standard Čech complexes $\check{C}^\bullet(\mathscr{U}, C^i), \delta)$ of the sheaf C^i, and whose horizontal differentials are given by d. Form the total complex of $(\check{C}^\bullet(\mathscr{U}, C^\bullet), d, \delta)$, i.e. the complex given by $\mathrm{Tot}^\bullet$ whose n-th term is $\mathrm{Tot}^n = \bigoplus_{i+j=n} \check{C}^i(\mathscr{U}, C^j)$ and whose differential $\mathrm{Tot}^n \to \mathrm{Tot}^{n+1}$ is $\delta + (-1)^n d$. Suppose that the covering $\mathscr{U}$ is "good", in the sense that

$$H^k(U_{i_1} \cap \cdots \cap U_{i_r}, C^i) = 0$$

for all $k, r = 1, 2, \ldots$ and $i = 0, 1, \ldots$. Then

$$\mathbb{H}^*(X, C^\bullet) \cong H^*(\mathrm{Tot}^\bullet)\,.$$

(For example, suppose X is a smooth projective variety of dimension n and we are considering a complex of quasi-coherent sheaves $C^\bullet$ on X. One can always find a cover by $n+1$ affine open subsets, and this cover is good because quasi-coherent sheaves are acyclic on affines.)

(a) Let X be a smooth projective variety over $\mathbb{C}$. Describe classes in $H^1_{\mathcal{D}}(X, \mathbb{Z}(1))$ using the above Čech description of hypercohomology.
(b) Describe the isomorphism $H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^0(X, \mathcal{O}_X^*)$ using part (a).
(c) Let X be a smooth projective variety over $\mathbb{C}$. Describe classes in $H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ using the above Čech description of hypercohomology.
(d) Describe the isomorphism $H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \cong \mathbb{H}^1(X, \mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X)$ using part (c).

3) Let X be a complex manifold. Show that the group $\mathbb{H}^1(X, \mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X)$ (which we know is isomorphic to $H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$) can be identified with the group of isomorphism classes of (holomorphic) line bundles $\mathscr{L}$ on X with a (holomorphic) connection $\nabla : \mathscr{L} \to \mathscr{L} \otimes_{\mathcal{O}_X} \Omega^1_X$. (Recall that this means that ∇ is a $\mathbb{C}$-linear map satisfying the Leibniz rule: $\nabla(fs) = f\nabla(s) + s \otimes df$ for local sections f of $\mathscr{L}$ and s of $\mathcal{O}_X$.)

4) Under the isomorphisms given in Lecture 2 (and also in the previous questions), the product map

$$\cup : H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \times H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \to H^1_{\mathcal{D}}(X, \mathbb{Z}(2))$$

becomes a map

$$r(-,-) : H^0(X, \mathcal{O}_X^*) \times H^0(X, \mathcal{O}_X^*) \to \mathbb{H}^1(X, \mathcal{O}_X^* \xrightarrow{d\log} \Omega_X^1)\,.$$

Let $f \in H^0(X, \mathcal{O}_X^*)$ such that $1 - f \in H^0(X, \mathcal{O}_X^*)$. Show that the line bundle with connection $r(f, 1-f)$ is trivial, i.e. $r(f, 1-f) \cong (\mathcal{O}_X, d)$.